

Quantum tunneling from scalar fields in rotating black strings

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ABSTRACT: Using the Hamilton-Jacobi method of quantum tunneling and complex path integration, we study Hawking radiation of scalar particles from rotating black strings. We discuss tunneling of both charged and uncharged scalar particles from the event horizons. For this purpose, we use the Klein-Gordon equation and find the tunneling probability of outgoing scalar particles. The procedure gives Hawking temperature for rotating charged black strings as well.

1. Introduction

Classically, black holes are perfect absorbers and do not radiate any particles. In the last forty years many advancements in the field of black hole physics came about as a result of the interplay between classical thermodynamics and quantum properties of black holes. In 1970's Bekenstein related the properties of black holes with the laws of thermodynamics [1]. Soon after this, Hawking showed that quantum mechanically black holes radiate particles [2, 3, 4]. This discovery was very important because it gave a new perspective to the quantum theory of gravity. Following this researchers showed a great interest in the field of black hole physics and used different methods to investigate these thermal radiations from black holes.

In 1990's Kraus and Wilczek [5, 6] developed the technique of studying Hawking radiations as a phenomenon of quantum tunneling. In this semi-classical approach, the imaginary part of the classical action is calculated for outgoing trajectories across the horizon. By using the WKB approximation, the tunneling probability for a classically forbidden trajectory of s wave coming from inside to outside the horizon is given by

$$\Gamma \propto \exp(-2\text{Im}I), \quad (1.1)$$

where I is the classical action of the trajectory to the leading order of Planck's constant, \hbar . If we compare this equation with $\Gamma = \exp(-\beta E)$, which is the Boltzmann factor, where E is the energy of the particle and β is the inverse temperature of the horizon, we can write Hawking temperature for a black hole.

Originally the tunneling method was applied to the Schwarzschild black hole [5]. However, this proved to be a powerful method and has been applied to a variety of black configurations ([7]-[17]) since then. There are two different ways to calculate the imaginary part of the classical action I for the emitted particle: the null geodesic method and the Hamilton-Jacobi ansatz. The first one was used by Parikh and Wilczek [9], which followed from the work of Kraus and Wilczek, and the second one, which is the extension of the complex path analysis [8, 10] has been used by different authors.

In this paper we use the Hamilton-Jacobi ansatz to study tunneling of scalar particles from cylindrically symmetric black holes, or black strings [18, 19]. For this purpose we will solve the Klein-Gordon equation both for the uncharged as well as the charged cases. Using WKB approximation and complex path integration we work out the tunneling probabilities of scalar particles across the event horizons. This method also recovers the correct value of Hawking temperature for rotating black strings. The paper is organised as follows. In the next section we explain the

metric describing rotating black strings. In Sections 3 and 4 we study tunneling of uncharged and charged scalars, respectively, from these objects. At the end we give a brief Conclusion.

2. Rotating black strings

We consider Einstein-Hilbert action in four dimensions with a cosmological constant in the presence of the electromagnetic field. Solving the Einstein-Maxwell equations for a charged rotating cylindrically symmetric spacetime gives [20, 21]

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{N(r)} - H(r)dtd\theta + K(r)d\theta^2 + L(r)dz^2, \quad (2.1)$$

where

$$F(r) = \alpha^2 r^2 - \frac{2G(M + \Omega)}{\alpha r} + \frac{4GQ^2}{\alpha^2 r^2}, \quad (2.2)$$

$$N(r) = \alpha^2 r^2 - \frac{2G(3\Omega - M)}{\alpha r} + \left(\frac{3\Omega - M}{\Omega + M}\right) \left(\frac{4GQ^2}{\alpha^2 r^2}\right), \quad (2.3)$$

$$H(r) = \frac{16GJ}{3\alpha r} \left(1 - \frac{2Q^2}{(\Omega + M)\alpha r}\right), \quad (2.4)$$

$$K(r) = r^2 + \frac{4G(M - \Omega)}{\alpha^3 r} \left(1 - \frac{2Q^2}{(\Omega + M)\alpha r}\right), \quad (2.5)$$

$$L(r) = \alpha^2 r^2. \quad (2.6)$$

Here M and Q are the ADM mass and charge of the black string, J is the angular momentum and $\Omega = \sqrt{M^2 - 8J^2\alpha^2/9}$, where $\alpha^2 = -\Lambda/3$, Λ being the cosmological constant. We can write Eq.(2.1) in another form [20]

$$ds^2 = -N^{02}dt^2 + R^2(N^\phi dt + d\theta)^2 + \frac{dr^2}{g(r)} + e^{-4\phi}dz^2, \quad (2.7)$$

where

$$N^{02} = \left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)^2 \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right) \frac{r^2}{R^2},$$

$$N^\phi = -\frac{\gamma\omega}{\alpha^2 R^2} \left(\frac{b}{\alpha r} - \frac{c^2}{\alpha r}\right),$$

$$R^2 = \gamma^2 r^2 - \frac{\omega^2}{\alpha^2} \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right),$$

$$g(r) = \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} \right),$$

$$e^{-4\phi} = \alpha^2 r^2,$$

and

$$b = 4GM \left(1 - \frac{3a^2\alpha^2}{2} \right),$$

$$c^2 = 4GQ \left(\frac{1 - \frac{3a^2\alpha^2}{2}}{1 - \frac{a^2\alpha^2}{2}} \right).$$

Here N^{0^2} and N^ϕ are the lapse and shift functions and a is the rotation parameter such that $a = J/M$. Further, γ^2 and ω^2/α^2 are defined as

$$\gamma^2 = \frac{2GM}{b} + \frac{2G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}} : \frac{\omega^2}{\alpha^2} = \frac{4GM}{b} - \frac{4G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}},$$

$$\gamma^2 = \frac{2GM}{b} - \frac{2G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}} : \frac{\omega^2}{\alpha^2} = \frac{4GM}{b} + \frac{4G}{b} \sqrt{M^2 - \frac{8J\alpha^2}{9}},$$

or

$$\gamma = \sqrt{\frac{1 - \frac{a^2\alpha^2}{2}}{1 - \frac{3a^2\alpha^2}{2}}}, \quad (2.8)$$

$$\omega = \frac{a\alpha^2}{\sqrt{1 - \frac{3a^2\alpha^2}{2}}}. \quad (2.9)$$

The line charge density along the z -line is given by

$$Q = \frac{Q_z}{\Delta z} = \gamma\lambda. \quad (2.10)$$

For the above line element the vector potential can be written as

$$A_t = -h(r)\delta_t^t = -\gamma h(r), \quad (2.11)$$

$$A_r = -h(r)\delta_t^r = 0, \quad (2.12)$$

$$A_\theta = -h(r)\delta_\theta^\theta = -\frac{\omega}{\alpha^2} h(r), \quad (2.13)$$

$$A_z = -h(r)\delta_t^z = 0, \quad (2.14)$$

where $h(r)$ is an arbitrary function of r .

3. Scalar particles from rotating black strings

To model the scalar tunneling from uncharged rotating black string we use the Klein-Gordon equation for a scalar field Ψ given by

$$g^{\mu\nu}\partial_\mu\partial_\nu\Psi - \frac{m^2}{\hbar^2}\Psi = 0. \quad (3.1)$$

We apply the WKB approximation and assume an ansatz of the form

$$\Psi(t, r, \theta, z) = e^{\left(\frac{i}{\hbar}I(t, r, \theta, z) + I_1(t, r, \theta, z) + O(\hbar)\right)}, \quad (3.2)$$

Now by using Eq. (3.2) in Eq. (3.1) in leading order of \hbar and dividing by the exponential term and multiplying by \hbar^2 , we get

$$g^{tt}(\partial_t I)^2 + g^{rr}(\partial_r I)^2 + g^{t\theta}\partial_t I\partial_\theta I + g^{\theta\theta}(\partial_\theta I)^2 + g^{zz}(\partial_z I)^2 + m^2 = 0. \quad (3.3)$$

The black string admits three Killing vectors $\langle \partial_t, \partial_\theta, \partial_z \rangle$. The existence of these symmetries implies that we can assume a solution for Eq. (3.3), in the form

$$I(t, r, \theta, z) = -Et + W(r) + J_1\theta + J_2z + K, \quad (3.4)$$

where E , J_1 , K and J_2 are constants and, further, we consider the radial trajectories only. Substituting Eq. (3.4) in Eq. (3.3) and solving for $W(r)$, we get

$$W_\pm(r) = \pm \int \frac{\sqrt{E^2 + n(r)(J_1)^2 - N^\phi(EJ_1) - w(r)(J_2)^2 - N^{02}m^2}}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)g(r)(r/R)} dr, \quad (3.5)$$

where

$$\begin{aligned} n(r) &= N^{\phi^2} - \frac{N^{02}}{R^2}, \\ w(r) &= -\frac{N^{02}}{\alpha^2 r^2}, \\ g(r) &= \alpha^2 r^2 - \frac{b}{\alpha r}. \end{aligned} \quad (3.6)$$

Noting that at $r = r_+$ we have a simple pole and, therefore, by using the residue theory for semi circle the integral yields

$$W_\pm(r) = \pm \pi i \frac{\sqrt{E^2 + n(r_+)(J_1)^2 - N^\phi(r_+)(EJ_1) - w(r_+)(J_2)^2}}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)g'(r_+)(r_+/R(r_+))}, \quad (3.7)$$

where

$$g'(r_+) = 2\alpha^2 r_+ + \frac{b}{\alpha r_+^2},$$

$$R(r_+) = \gamma r_+.$$

From Eq. (3.7) we see that

$$ImW_{\pm}(r) = \pm\pi \frac{\gamma \sqrt{E^2 + n(r_+)(J_1)^2 - N^\phi(r_+)(EJ_1) - w(r_+)(J_2)^2}}{(\gamma^2 - \frac{\omega^2}{\alpha^2}) g'(r_+)}, \quad (3.8)$$

Now, the probabilities of crossing the horizon from inside to outside and outside to inside are given by [8, 10]

$$P_{emission} \propto \exp\left(\frac{-2}{\hbar} ImI\right) = \exp\left(\frac{-2}{\hbar} (ImW_+ + ImK)\right), \quad (3.9)$$

$$P_{absorption} \propto \exp\left(\frac{-2}{\hbar} ImI\right) = \exp\left(\frac{-2}{\hbar} (ImW_- + ImK)\right). \quad (3.10)$$

As the probability of any incoming particles crossing the horizons have a 100% chance of entering the black hole, therefore, it is necessary to set

$$ImK = -ImW_-, \quad (3.11)$$

in the above equation. From Eq. (3.7), we get

$$W_+ = -W_-. \quad (3.12)$$

This means that the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma = \exp\left(-\frac{4}{\hbar} ImW_+\right). \quad (3.13)$$

From Eq. (3.8), putting the value of ImW_+ in Eq. (3.13), we get

$$\Gamma = \exp\left(-4\pi \frac{\gamma \sqrt{E^2 + n(r_+)(J_1)^2 - N^\phi(r_+)(EJ_1) - w(r_+)(J_2)^2}}{(\gamma^2 - \frac{\omega^2}{\alpha^2}) g'(r_+)}\right). \quad (3.14)$$

This is the probability of an outgoing scalar particle from the event horizon of rotating black string. From this we can find Hawking temperature for uncharged rotating black string by comparing with the Boltzmann factor [8, 10], $\Gamma = \exp(-\beta E)$, where

E is the energy of particle and β is the inverse temperature. Thus the Hawking temperature takes the form

$$T_H = \frac{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g'(r_+)}{4\pi\gamma} \quad (3.15)$$

or

$$T_H = \frac{1}{4\pi\gamma} \left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) \left(2\alpha^2 r_+ + \frac{b}{\alpha r_+^2}\right). \quad (3.16)$$

By using Eqs. (2.8) and (2.9), we note that $\gamma^2 - (\omega^2/\alpha^2) = 1$. So the temperature becomes

$$T_H = \frac{1}{4\pi\gamma} \left(2\alpha^2 r_+ + \frac{b}{\alpha r_+^2}\right). \quad (3.17)$$

4. Scalar particles from charged rotating black strings

To study the contribution of scalar particles towards Hawking radiation from charged rotating black strings, we use the charged Klein-Gordon equation for scalar field $\Psi(t, r, \theta, z)$

$$\frac{1}{\sqrt{-g}} \left(\partial_\mu - \frac{iq}{\hbar} A_\mu\right) \left(\sqrt{-g} g^{\mu\nu} (\partial_\nu - \frac{iq}{\hbar} A_\nu) \Psi\right) - \frac{m^2}{\hbar^2} \Psi = 0. \quad (4.1)$$

Following a procedure similar to that of the previous section, we let

$$\Psi(t, r, \theta, z) = e^{\left(\frac{i}{\hbar} I(t, r, \theta, z) + I_1(t, r, \theta, z) + O(\hbar)\right)}. \quad (4.2)$$

Taking summation on μ and ν in Eq. (4.1) and using Eq. (4.2) in leading order of \hbar , we get the differential equation of the form

$$\begin{aligned} &g^{tt}(\partial_t I - qA_t)^2 + g^{rr}(\partial_r I)^2 + g^{t\theta}(\partial_t I \partial_\theta I - 2qA_t \partial_\theta I + q^2 A_t A_\theta) \\ &+ g^{\theta\theta}(\partial_\theta I - qA_\theta)^2 + g^{zz}(\partial_z I)^2 + m^2 = 0. \end{aligned} \quad (4.3)$$

By assuming a solution of the form in Eq. (3.4) for the above and evaluating for $W(r)$ gives

$$\begin{aligned} W_\pm(r) = \pm \int \frac{dr}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g(r)(r/R)} \left[(E + qA_t)^2 + n(r)(J_1 - qA_\theta)^2 - \right. \\ \left. N^\phi(EJ_1 + 2qA_t J_1 + q^2 A_t A_\theta) - w(r)(J_2)^2 - N^{02} m^2 \right]^{1/2}, \end{aligned} \quad (4.4)$$

where

$$\begin{aligned} n(r) &= N^{\phi^2} - \frac{N^{0^2}}{R^2}, \\ w(r) &= -\frac{N^{0^2}}{\alpha^2 r^2}, \\ g(r) &= \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}. \end{aligned}$$

Here, we have a simple pole at $r = r_+$, and thus, by the residue theory we evaluate the integral as

$$\begin{aligned} W_{\pm}(r) &= \pm \frac{\pi i \gamma}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g'(r_+)} \left[(E + qA_t)^2 + n(r_+)(J_1 - qA_{\theta})^2 - \right. \\ &\quad \left. N^{\phi}(r_+)(EJ_1 + 2qA_tJ_1 + q^2A_tA_{\theta}) \right]^{1/2}, \end{aligned} \quad (4.5)$$

where

$$g'(r_+) = 2\alpha^2 r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2 r_+^3}. \quad (4.6)$$

This implies that

$$\begin{aligned} ImW_{\pm}(r) &= \pm \frac{\pi \gamma}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g'(r_+)} \left[(E + qA_t)^2 + n(r_+)(J_1 - qA_{\theta})^2 - \right. \\ &\quad \left. N^{\phi}(r_+)(EJ_1 + 2qA_tJ_1 + q^2A_tA_{\theta}) \right]^{1/2}. \end{aligned} \quad (4.7)$$

Thus the probability of a particle tunneling from inside to outside the horizon as given by Eq. (3.13) on substituting the value of ImW_+ from the above equation takes the form

$$\begin{aligned} \Gamma &= \exp \left(\frac{-4\pi \gamma}{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g'(r_+)} \left[(E + qA_t)^2 + n(r_+)(J_1 - qA_{\theta})^2 - \right. \right. \\ &\quad \left. \left. N^{\phi}(r_+)(EJ_1 + 2qA_tJ_1 + q^2A_tA_{\theta}) \right]^{1/2} \right). \end{aligned} \quad (4.8)$$

We can find the Hawking temperature for rotating charged black string by comparison with the Boltzmann factor as before. So

$$T_H = \frac{1}{4\pi} \frac{\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) g'(r_+)}{\gamma}, \quad (4.9)$$

or

$$T_H = \frac{1}{4\pi\gamma} \left(2\alpha^2 r_+ + \frac{b}{\alpha r_+^2} - \frac{2c^2}{\alpha^2 r_+^3} \right), \quad (4.10)$$

where we have used $\gamma^2 - (\omega^2/\alpha^2) = 1$. This formula is consistent with the literature [17, 22].

5. Conclusion

Hawking radiations from black holes comprise the whole range of spectrum of particles including fermions, bosons, gravitinos etc. In particular, emission of scalar fields has been studied for various spherically symmetric black holes. In this paper we have extended this analysis to cylindrically symmetric and rotating black configurations. Using the so-called Hamilton-Jacobi method and the complex path integration technique, we have solved the Klien-Gordon equation in the background of charged rotating black strings. Employing WKB approximation we have found the tunneling probabilities of uncharged and charged particles crossing the event horizon. An important consequence of this procedure is that we get the correct value of Hawking temperature. The temperature for charged rotating black strings is given in Eq. (4.10). If we put angular momentum equal to zero, the temperature reduces to that for the charged non rotating black strings [16]. Taking charge Q to be zero gives temperature for the uncharged case as

$$T = \frac{1}{2\pi} \left(\alpha^2 r_+ + \frac{2M}{\alpha r_+^2} \right). \quad (5.1)$$

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